$$\frac{dn_\mu}{dt} = k_+ C(\vec{X}_\mu, t)\,[1 - n_\mu] - k_- n_\mu \qquad ①$$

$$\frac{\partial C}{\partial t} = D\nabla^2 C - \sum_{\mu=1}^{N} \delta(\vec{X} - \vec{X}_\mu)\,\frac{dn_\mu}{dt} \qquad ②$$

To study fluctuations around the steady state solution we write

$$n_\mu(t) = \bar{n}_\mu + \delta n_\mu \quad ; \quad C(\vec{x}, t) = \bar{C} + \delta C(\vec{x}, t) \qquad ③$$

where barred quantities reffer to steady state solutions of ① and ②. We substitute ③ in ① and ② keeping only terms that are linear in δn_μ and δc yields

$$\frac{d\delta n_\mu}{dt} = -\bar{k}_+ \bar{C}\,\delta n_\mu + \bar{k}_+ (1 - n_\mu)\,\delta c(\vec{x}_\mu, t) + \bar{C}(1 - \bar{n}_\mu)\,\delta k_+ - \bar{k}_- \delta n_\mu - \bar{n}_\mu \delta k_- \qquad ④$$

$$\frac{\partial \delta c}{\partial t} = D\nabla^2 \delta c - \sum_{\nu=1}^{N} \delta(\vec{X} - \vec{X}_\nu)\,\frac{d\delta n_\nu}{dt} \qquad ⑤$$

The general solution for may be written as

$$\delta n_\mu(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \delta\tilde{n}_\mu(\omega)\, e^{-i\omega t}\, d\omega$$

$$\delta C(\vec{x}, t) = \frac{1}{(2\pi)^4} \int d^3\vec{k} \int_{-\infty}^{+\infty} d\omega\; \delta\tilde{C}(\vec{k}, \omega)\, e^{-i(\omega t - \vec{k}\cdot\vec{x})}$$

where $\delta\tilde{n}_\mu(\omega)$ and $\delta\tilde{C}(\vec{k}, \omega)$ are the solutions of ④ and ⑤ in Fourier space which is spanned by 3-D vector $\vec{k}$ and ω

If we substitute $\delta n_\mu(t)$ and $\delta C(\vec{x}, t)$ in (4) and (5) we get:

(6) $$(-i\omega + \bar{k}_- + \bar{k}_+ \bar{c})\,\delta\tilde{n}_\mu = \bar{k}_+ (1-\bar{n}_\mu)\,\delta\tilde{C}(\vec{X}_\mu, \omega) + \frac{\bar{k}_+ \bar{c}(1-\bar{n}_\mu)}{k_B T}\,\delta\tilde{F}(\omega)$$

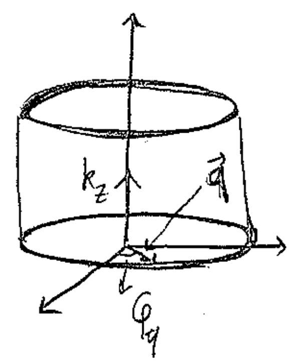
(7) $$(D\vec{k}^2 - i\omega)\,\delta\tilde{c}(\vec{k},\omega) = \sum_{\nu=1}^{N} i\omega\,\delta\tilde{n}_\nu(\omega)\, e^{+i\vec{k}\cdot\vec{X}_\nu}$$

we can rewrite (7) as

$$(D\vec{k}^2 - i\omega)\,\delta\tilde{C}(\vec{k},\omega) = i\omega\,\delta\tilde{n}_\mu(\omega)\, e^{+i\vec{k}\cdot\vec{X}_\mu} + i\omega \sum_{\nu\neq\mu}^{N} \delta\tilde{n}_\nu(\omega)\, e^{+i\vec{k}\cdot\vec{X}_\nu} \Rightarrow$$

It is easy to see that Fourier transform of $\delta\tilde{C}$ over $\vec{k} \Rightarrow$

(8) $$\delta\tilde{C}(\vec{X}_\mu, \omega) = i\omega\,\delta\tilde{n}_\mu(\omega)\,\frac{1}{(2\pi)^3}\int \frac{d^3k}{Dk^2 - i\omega} + \frac{i\omega}{(2\pi)^3}\sum_{\nu\neq\mu}^{N}\int \frac{d^3k}{Dk^2 - i\omega}\, e^{-i\vec{k}\cdot(\vec{X}_\mu - \vec{X}_\nu)}$$



The first integral on the r.h.s. is

$$\int \frac{d^3k}{Dk^2 - i\omega} = \frac{1}{D}\int_0^\infty q\,dq \int_0^{2\pi} d\varphi_q \int_{-\infty}^{+\infty} \frac{dk_z}{k_z^2 + q^2 - \frac{i\omega}{D}}$$

$$= \frac{2\pi}{D}\int_0^\infty q\,dq\, \frac{\pi}{\sqrt{q^2 - i\frac{\omega}{D}}}$$

This integral diverges as $q \to \infty$. This divergence is due to the fact that we modeled the receptors by Delta function. To resolve this we introduce an upper cutoff Λ on the momentum q. The cutoff $\Lambda \approx \frac{\pi}{a}$ where a is the linear size of the receptor $\Rightarrow$

$$\int \frac{d^3k}{Dk^2 - i\omega} = \frac{2\pi^2}{D} \sqrt{q^2 - \frac{i\omega}{D}} \Big|_0^{\Lambda} = \frac{2\pi^2}{D} \left(\sqrt{\Lambda^2 - \frac{i\omega}{D}} - \sqrt{-\frac{i\omega}{D}} \right)$$

for very small frequency $\Lambda \gg \sqrt{\frac{\omega}{D}} \Rightarrow$

$$\int \frac{d^3k}{Dk^2 - i\omega} \approx \frac{2\pi^3}{Da}$$

The second integral in (8) does not diverge and its value in the limit of very low frequency is

$$\int_0^{\infty} q\,dq \int_0^{2\pi} d\varphi_q \int_{-\infty}^{+\infty} \frac{e^{-\vec{k}\cdot(\vec{X}_\mu - \vec{X}_\nu)}}{Dk^2 - i\omega} dk_z \approx \frac{2\pi^2}{D} \frac{1}{|\vec{X}_\mu - \vec{X}_\nu|}$$

$\Rightarrow$

(9) $$\delta C(\vec{X}_\mu, \omega) = \frac{i\omega}{4\pi D a} \delta \tilde{n}_\mu + \frac{i\omega}{4\pi D} \sum_{\nu \neq \mu}^{N} \frac{\delta \tilde{n}_\nu(\omega)}{|\vec{X}_\mu - \vec{X}_\nu|}$$

We substitute (9) in (6) we find:

$$\left[-i\omega\left(1 + \frac{\bar{k}_+(1-\bar{n}_\mu)}{4\pi D a}\right) + (k_- + k_+\bar{c}) \right] \delta\tilde{n}_\mu = \frac{i\omega \bar{k}_+(1-\bar{n}_\mu)}{4\pi D} \sum_{\nu \neq \mu}^{N} \frac{\delta \tilde{n}_\nu}{|\vec{X}_\mu - \vec{X}_\nu|} + \frac{k_+ \bar{c}(1-\bar{n}_\mu)}{k_B T} \delta\tilde{F}(\omega)$$

I will defined

$$\alpha = 1 + \frac{\bar{k}_+(1-\bar{n}_\mu)}{4\pi D a} \quad \text{Then}$$

$$\delta\tilde{n}_\mu(\omega) = \frac{i\omega\bar{k}_+(1-\bar{n}_\mu)}{4\pi D\left[(k_- + k_+\bar{c}) - i\alpha\omega\right]} \sum_{\nu\neq\mu}^{N} \frac{\delta\tilde{n}_\nu(\omega)}{|\vec{X}_\mu - \vec{X}_\nu|}$$

$$+ \frac{k_+\bar{c}(1-\bar{n}_\mu)}{k_B T} \frac{\delta\tilde{F}(\omega)}{\left[(\bar{k}_- + \bar{k}_+\bar{c}) - i\alpha\omega\right]}$$

If we consider all receptors on the bacterium body are

$$\delta\tilde{n}(\omega) = \sum_{\mu=1}^{N} \delta\tilde{n}_\mu(\omega)$$

then

$$\delta\tilde{n}(\omega) = \frac{i\omega\bar{k}_+(1-\bar{n}_\mu)}{4\pi D\left[(k_- + k_+\bar{c}) - i\alpha\omega\right]} \sum_{\mu=1}^{N}\sum_{\nu\neq\mu}^{N} \frac{\delta\tilde{n}_\nu(\omega)}{|\vec{X}_\mu - \vec{X}_\nu|}$$

$$+ \frac{N k_+\bar{c}(1-\bar{n}_\mu)}{k_B T} \frac{\delta\tilde{F}(\omega)}{\left[(k_- + k_+\bar{c}) - i\alpha\omega\right]}$$

Because of the assumption that all N-receptors are identical then the double sum can be simplified

$$\sum_{\mu=1}^{N}\sum_{\nu\neq\mu}^{N} \frac{\delta\tilde{n}_\nu(\omega)}{|\vec{X}_\mu - \vec{X}_\nu|} = \delta\tilde{n} \sum_{\nu=2}^{N} \frac{1}{|\vec{X}_1 - \vec{X}_\nu|}$$

Putting it all together we find:

$$\delta\tilde{n}(\omega) = \frac{i\,\omega\bar{k}_+(1-\bar{n}_1)\delta\tilde{n}}{4\pi D\left[(k_- + k_+\bar{c}) - i\alpha\omega\right]} \sum_{\nu=2}^{N} \frac{1}{|\vec{X}_\nu - \vec{X}_1|}$$

$$+ \frac{N k_+\bar{c}(1-\bar{n}_1)}{k_B T} \frac{\delta\tilde{F}}{\left[(k_- + k_+\bar{c}) - i\alpha\omega\right]} \Rightarrow$$

$$\left[1 - \frac{i\omega\bar{k}_+(1-\bar{n}_1)}{4\pi D\left[(\bar{k}_- + \bar{k}_+\bar{c}) - i\alpha\omega\right]}\phi(N)\right]\delta\tilde{n}(\omega) =$$

$$\frac{N k_+\bar{c}(1-\bar{n}_1)}{k_B T} \frac{\delta\tilde{F}}{\left[(\bar{k}_- + \bar{k}_+\bar{c}) - i\alpha\omega\right]}$$

where $$\phi(N) = \sum_{\nu=2}^{N} \frac{1}{|X_\nu - X_1|}$$